Cyclic base orders of matroids

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Abstract

This is a typewritten version, with many corrections, of a handwritten note, August 1984, for a course taught by Jack Edmonds. The purpose is to show the origin of what the author now calls the "opposing chevron problem", in section 4. The problem is whether any nonsingular square matrix over a field can have its rows and columns reordered so that all leading principal minors and their complements are nonzero.

1 The Problem

Given an n-dimensional vector space $V = V(n, q)$ over a finite field F of order q, the nonzero elements of V may be arranged in a cycle $v_1, v_2, ..., v_{q^n-1}$ such that any n consecutive members of this cycle form a base of V. To do this a standard construction from coding theory is used. There is a field E of order q^n, the elements of which form an n-dimensional vector space over the subfield F. Every finite field has a multiplicative generator, meaning there is an $x \in E$ such that $\{x, x^2, ..., x^{q^n-1}\} = E\backslash\{0\}$. This gives the required cycle because for any i, $\{x^i, x^{i+1}, \ldots, x^{i+n-1}\}$ forms a base. This is because any linear dependency of the form $c_i x^i + c_{i+1}x^{i+1} + \ldots + c_{i+n-1}x^{i+n-1} = 0$ with all c_j in F would imply, after division by an appropriate power of x, that $x^k = d_0x^0 + \ldots + d_{k-1}x^{k-1}$ for some $k < n$, $d_j \in F$. Then <u>any</u> power of x would be a linear combination of the first k powers of x and this contradicts that E has dimension n.

A <u>goal</u> is to show that constructions like this can be produced using facts about <u>matroids</u>, thus generalizing the construction.

2 Straight-Line Order Is Easy

As a general problem we would like to know if the elements of a matroid, M, can be cyclically ordered such that any n consecutive elements form a base.

Definition. A "basic cyclic order" of the elements of a matroid is an ordering of all its elements $e_1, \ldots, e_m$ such that any n consecutive (indices modulo m) form a base, where n is the rank of the matroid.

For technical simplicity we will assume that the rank, n, of the matroid divides the number of its elements, $|M|$. If such a cyclic ordering exits it must be possible to partition the elements of M into $\ell = |M|/n$ bases by taking disjoint groups of n around the circle. A theorem of Edmonds [[1], 8.4] allows us to tell if M can be partitioned into bases. Given such a partition we can attempt to order the bases $B_1, \ldots, B_\ell$ such that the elements of each B_i can be ordered to make the whole list a basic cyclic order. As far as I know it may be possible to produce a basic cyclic order from any cyclic ordering of a partition into bases. If the end around part of the constraint is removed it is very easy to produce "linear basic orders".

Proposition. If $B_1, \ldots B_\ell$ are any ℓ bases of a matroid then for any ordering $b_1^1, b_2^1, \ldots, b_n^1$ of the elements of B_1 there is an ordering of each $B_i, i > 1$ as $b_1^i, \ldots, b_n^i$ such that every n consecutive elements of the linear list

$$b_1^1, \ldots, b_n^1, b_1^2, \ldots, b_n^2, \ldots, b_1^\ell, \ldots, b_n^\ell$$

is a base.

Proof. The statement will be proved for $\ell = 2$ and then it will follow for arbitrary ℓ by ordering B_3 after fixing the order on B_2. Then B_4 can be ordered, and so on.

Note $r(\{b_2^1, b_3^1, \ldots, b_n^1\} \cup B_2) = r(B_2) = n$, so the independent set $\{b_2^1, \ldots, b_n^1\}$can be extended by some $b_1^2 \in B_2$ to form a base. Then since $r(\{b_3^1, \ldots, b_n^1\} \cup B_2) = n$ and $\{b_3^1, \ldots, b_n^1, b_1^2\}$ is independent there is an extension by b_2^2 in $B_2 \backslash \{b_1^2\}$ which forms a base, and so on until all of B_2 has been ordered. $\square$

3 A Circular Exchange Property

Since the above proof that any base partition can be linearly ordered is "too easy" there is reason to believe that any base partition can be cyclically ordered. We might hope that some exchange theorem will allow us to produce the circular order. For example, there is the theorem (communicated by U.S.R. Murty) that given any two bases of a matroid, say B and C and any partition of B as $X \cup Y$ there is a partition of C as $X' \cup Y'$ such that $X \cup Y'$ and $X' \cup Y$ are both bases. This means that the union of any two consecutive members of the circular order $XYX'Y'$ form a base. It seems difficult to refine these partitions down to individual elements. However we can prove the following extension of the previously mentioned theorem.

Theorem. If $B_1, \ldots, B_\ell$ are ℓ bases of a matroid then if $B_1 = X_1 \cup Y_1$ is any partition of B_1 then there exist partitions $B_i = X_i \cup Y_i$ of each B_i, $2 \leq i \leq \ell$, such that each $Y_i \cup X_{i+1}$ is base (indices taken modulo ℓ).

Proof. Let M be the matroid in question. Let

$$M^{\cup} = M/Y_1 \,\dot\cup\, M_3 \,\dot\cup\, M_4 \,\dot\cup \cdots \dot\cup\, M_\ell \,\dot\cup\, M/X_1 \tag{1}$$

where each M_i is a copy of M. We assume $\ell \geq 2$. For ease of notation let $M_2 = M/Y_1$ and $M_1 = M/X_1$. Form a generalized matching matroid M^+ on the set

$$B_2 \,\dot\cup\, B_3 \,\dot\cup \cdots \dot\cup\, B_l \tag{2}$$

where there is an edge from each element of B_i to the corresponding element in M_i and to the corresponding element in M_{i+1} (indexed modulo ℓ). If M_i or M_{i+1} is a contraction the edge goes to the image of that element under the contraction. The important observation is that the requirments of the theorem are satisfied iff

$$r(M^+) = |M^+| = (\ell - 1)n. \tag{3}$$

If the rank is this large there is a matching involving all elements of M^+. In such a matching let X_i be the subset of B_i matched to elements in M_i and let Y_i be those matched to M_{i+1} and we have the required partitions.

By the Edmonds' theorem we know that if (3) does not hold there is a set S of elements of M^+ whose neighborhood in $M^{\cup}$ has rank $< |S|$. Let $S_i = S \cap B_i$. Assume then that

$$r_{M/Y_1}(S_2) + r_M(S_2 \cup S_3) + \cdots + r_M(S_{\ell-1} \cup S_\ell) + r_{M/X_1}(S_\ell) < |S_2| + \cdots + |S_\ell|. \tag{4}$$

Let r denote r_M. Now

$$[r(Y_1 \cup S_2) - r(S_2)] + [r(S_2 \cup S_3) - r(S_3)] = r(Y_1 \cup S_2) + r(S_2 \cup S_3) - r(S_2) - r(S_3)$$

and by rank submodularity this is $\geq$

$$r(Y_1 \cup S_3) + r(S_2) - r(S_2) - r(S_3) = r(Y_1 \cup S_3) - r(S_3).$$

Next

$$[r(Y_1 \cup S_2) - r(S_2)] + [r(S_2 \cup S_3) - r(S_3)] + [r(S_3 \cup S_4) - r(S_4)]$$

$$\geq [r(Y_1 \cup S_3) - r(S_3)] + [r(S_3 \cup S_4) - r(S_4)] \geq [r(Y_1 \cup S_4) - r(S_4)],$$

by applying rank submodularity again. Continuing this way we obtain

$$[r(Y_1 \cup S_2) - r(S_2)] + [r(S_2 \cup S_3) - r(S_3)] + \cdots + [r(S_{\ell-1} \cup S_\ell) - r(S_\ell)] + [r(S_\ell \cup X_1) - r(X_1)]$$

$$\geq r(Y_1 \cup X_1) - r(X_1).$$

Note $r(Y_1 \cup X_1) - r(X_1) = |Y_1|$ and $r(S_i) = |S_i|$. Making these substitutions and rearranging terms,

$$[r(Y_1 \cup S_2) - |Y_1|] + r(S_2 \cup S_3) + \cdots + r(S_{\ell-1} \cup S_\ell) + [r(X_1 \cup S_\ell) - |X_1|]$$

$$\geq |S_2| + \cdots + |S_\ell|,$$

contradicting (4). $\square$

4 The Two Base Case

Hopefully when the matroid can be partitioned into two bases it would be easier to decide if a basic cyclic order exists, but I have not solved even this. Let M be a matroid which is represented as vectors over a field and let A and B be two bases. Without loss of generality we may assume A consists of n unit vectors. Then let A be represented by the $n \times n$ identity matrix and B as an $n \times n$ matrix whose columns are the vectors. A and B can each be reordered to form a basic cyclic ordering iff there is a column permutation of B, say B' and a permutation matrix Π such that any n consecutive columns of $[\Pi|B']_{n\times 2n}$ (cyclic indices) are independent. The rows of this may be rearranged to form $[I|B'']_{n\times 2n}$. The condition that any n consecutive columns of this form a base is exactly that all [leading] principal minors and all anti-principal minors of B'' are nonsingular. By an anti-principal minor we mean a square submatrix with corner at the lower right of B''.

Thus, for linearly representable matroids, any two bases have a basic cyclic order iff every nonsingular square matrix B can be converted by row and column permutations into a matrix B'' with all leading principal minors and their complements nonsingular. I don't know if this can always be done.

5 Two Spanning Trees

After reading a paper by Farber, Richter and Shank ("edge-disjoint Spanning Trees: A Connectedness Theorem") it became clear that the method of proof in that paper could be used to prove the following theorem which shows there is a basic cyclic order for any two trees.

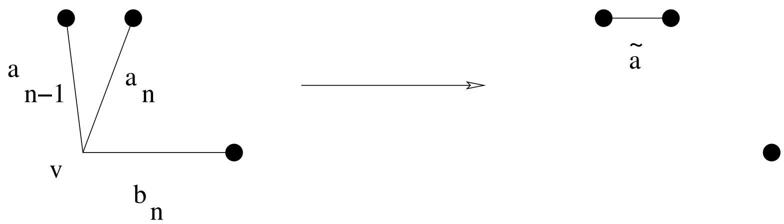
Theorem. If A and B are any two spanning trees of a graph G, then A and B can be ordered, $A = \{a_1, \ldots, a_n\}$ $B = \{b_1, \ldots, b_n\}$ such that any n cyclically consecutive elements of $a_1a_2\ldots a_nb_1\ldots b_n$ form a spanning tree of G.

Proof. We use mathematical induction. The statement is obviously true for graphs with two vertices (n=1). Assume now the statement is true whenever the trees have size $n-1$. All edges of G can be neglected except the A and B edges. Moreover, it can be assumed that $A \cap B = \emptyset$ because any common edge can be replaced by a double edge - one part in A and the other in B.

The main point is that any such graph must have a vertex of valency 2 or 3. G has $n+1$ vertices and the sum of all valencies is twice the number of edges or $4n$, so not all vertices can have valency at least 4. If there is a vertex v of degree 2 it is incident with one A-edge and one B-edge, say a_n and b_n. Using the induction hypothesis on $G-v$, there is a basic cyclic order $a_1\ldots a_{n-1}b_1\ldots b_{n-1}$ for $G-v$. Now it is easily seen that putting a_n anywhere in the A-sequence and b_n in the corresponding place in the B-sequence produces a basic cyclic order of G. For example, $a_1\ldots a_{n-1}a_nb_1\ldots b_{n-1}b_n$ is such an order.

If there is a vertex v of degree 3 then without loss of generality assume it is incident with two A-edges and one B-edge, say a_{n-1}, a_n and b_n. Remove v

from G and add an edge $\widetilde{a}$ connecting the ends of a_{n-1} and a_n as shown, to make the graph $\widetilde{G}$.



Now using the inductive hypothesis there is a basic cyclic order of $\widetilde{G}$, [Written on two lines.]

$$a_1 \dots a_{i-1} \; \widetilde{a} \; a_i \dots a_{n-2} \dots$$
$$b_1 \dots b_{i-1} \; b_i \; b_{i+1} \dots b_{n-1}.$$

Let b_i appear opposite $\widetilde{a}$. To produce a cyclic order for G, insert b_n immediately before b_i. We claim replacing $\widetilde{a}$ by the pair a_{n-1}, a_n in some order produces a basic cyclic order for G. Any spanning tree of $\widetilde{G}$ not containing $\widetilde{a}$ becomes a spanning tree of G when b_n, a_n or a_{n-1} are added. Also, any spanning tree of $\widetilde{G}$ containing $\widetilde{a}$ becomes a spanning tree of G when $\widetilde{a}$ is replaced by a_{n-1} and a_n.

To decide which order to use on the pair $\{a_{n-1}, a_n\}$, it must be shown that one of

$$a_{n-1}[a_i \dots a_{n-2} b_1 \dots b_{i-1}] b_n, \text{or}$$

$$a_n[a_i \dots a_{n-2} b_1 \dots b_{i-1}] b_n$$

is a tree in G. Notice $a_{n-1} a_n [a_i \dots a_{n-2} b_1 \dots b_{i-1}]$ is a tree in G. Adding the edge b_n produces a fundamental cycle of this tree and this and this cycle must pass through v. Therefore the cycle must use one and only one of $\{a_{n-1}, a_n\}$. Deleting this one produces the desired tree. □

References

[1] D.J.A. Welsh, *Matroid Theory*, Academic Press, 1976.