How Euler Proved the Pentagonal Expansion

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Abstract

The author wishes to write down concisely the original proof Euler gave of his pentagonal product formula.

1 Euler's Proof

The problem which Euler solved is to find the Taylor series expansion of $p = (1-x)(1-x^2)(1-x^3)\cdots$. The initial breakthough Euler made was to realize that

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\cdots = 1-\alpha-\beta(1-\alpha)-\gamma(1-\alpha)(1-\beta)-\delta(1-\alpha)(1-\beta)(1-\gamma)-\cdots.$$

This can be seen by grouping terms in the expansion according to their alphabetically last letter.

Apply this and use the shorthand $y_i = (1-x^i)$.

$$p = (1-x)(1-x^2)(1-x^3)\cdots = 1-x-x^2y_1-x^3y_1y_2-x^4y_1y_2y_3-\cdots.$$

Note $p = 1-x-x^2A_1$, where

$$A_1 = y_1 + xy_1y_2 + x^2y_1y_2y_3 + \cdots.$$

Consider any two consecutive terms in the series for A_1,

$$x^{k-1}y_1y_2\cdots y_k + x^ky_1y_2\cdots y_ky_{k+1}.$$

The second key idea is to realize that a simplification occurs when we replace y_1 with $1-x$ and collect equal powers of x. Each term is going to expand into two terms corresponding to the 1 and $-x$ parts. The point is that the $-x$ portion of the first term nearly cancels with the 1 portion the second term. The result is $-x^{2k+1}y_2\cdots y_k$. Note that to get this result we had to expand $y_{k+1} = 1-x^{k+1}$, but no other y_i had to be expanded.

Writing down this result for each pair and noting that the first part of the first term is uncanceled,

$$A_1 = 1 - \sum_{k \geq 1} x^{2k+1} y_2 \cdots y_k,$$

so

$$A_1 = 1 - x^3 - x^5 A_2,$$

where

$$A_2 = \sum_{k \geq 0} x^{2k} y_2 \cdots y_{k+2}.$$

A remarkable fact is that if we now replace the y_2 in each term with $1 - x^2$ and expand, we again get near cancellation, again expanding only the first and last y_i in each term.

$$A_2 = (1 - x^2) + x^2(1 - x^3) - x^8(y_3 + x^3 y_3 y_4 + x^6 y_3 y_4 y_5 + \cdots).$$

Of course, we want to define A_3 to be this last term. In general, $A_j = \sum_{k \geq 0} x^{j \cdot k} y_j \cdots y_{j+k}$.. The same trick always works and we have

$$A_j = 1 - x^{2j+1} - x^{3j+2} A_{j+1}.$$

We finally finish the result by rewriting p in terms of A_j with larger and larger j. Thus,

$$\begin{aligned} p &= 1 - x - x^2 A_1 \\ &= (1 - x) - x^2(1 - x^3) + x^7 A_2. \end{aligned}$$

In general, $p = q_j(x) + (-1)^j x^{n_j} A_j$, where $n_j = 2 + 5 + 8 + \cdots + (3j + 2)$. Of course, letting $j \to \infty$, $q_j(x) \to p(x)$. So,

$$\begin{aligned} p(x) &= 1 - x + \sum_{j>0} (-1)^j x^{\frac{3j^2+j}{2}} (1 - x^{2j+1}) \\ &= 1 + \sum_{j>0} (-1)^j (x^{\frac{3j^2-j}{2}} + x^{\frac{3j^2+j}{2}}) \end{aligned}$$

which is Euler's result.

References

[1] Jordan Bell, *Euler and the pentagonal number theorem*, preprint, 2006.